UNSTABLE RELATIONSHIPS*

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Abstract

We analyze models where agents search for partners to form relationships (employment, marriage, etc.) and may continue searching for different partners while matched. Matched agents are less inclined to search if their match yields more utility, and also if it is more stable. If one partner searches the relationship is less stable, so the other is more inclined to search, potentially making instability a self-fulfilling prophecy. We show this can generate a multiplicity – indeed, a continuum – of steady state equilibria. Whether there are multiple equilibria or a unique equilibrium, there tends to be excessive turnover, unemployment, and inequality compared to the efficient outcome.

Keywords: search, matching, marriage, unemployment, inequality

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1 Introduction

We analyze situations where agents search for partners to form bilateral relationships, such as marriage or employment relationships, with the following key feature: while matched, an individual may at some cost continue to search for a different partner. Matched agents are less inclined to search – that is, more inclined to be "faithful" to their current partner – if the match is better in the sense of instantaneous utility, and also if the match is more stable. What lends stability to a relationship? If your partner is searching, your relationship is less secure because you are more likely to be abandoned, and so you will be more inclined to search in this relationship. In this simple way, instability can be a self-fulfilling prophecy. We show these considerations can lead to multiple equilibria, and indeed to a continuum of steady state equilibrium in one version of the model. We also study the implications for efficiency. The model predicts that there tends to be inefficiently high levels of turnover, unemployment, and inequality in equilibrium.

The source of multiplicity and inefficiency here is new. For instance, it has nothing to do with the standard thick-market effects that have been understood in equilibrium search theory since Diamond (1982). The usual thick-market effect works as follows: assuming increasing returns in the matching technology, if there is more search then it is easier to meet people, and this makes people more inclined to search. To emphasize the distinction, for most of the paper we assume constant returns to scale in the meeting technology, so that your probability of meeting a potential partner is independent of aggregate search behavior, and make some other assumptions to reduce strategic interactions between market activity and individual decisions. This allows us to focus on the strategic interactions within relationships, and especially on endogenous instability. It is this feature that generates a continuum of steady state equilibria in the general version of the model, while the standard search externalities in the literature imply that steady states are locally unique.

Our basic framework assumes that when any two agents meet they draw a random x describing the instantaneous utility that each will receive if they form a partnership (i.e., individuals are homogeneous ex ante, but relationships are heterogeneous ex post due to match-specific idiosyncracies). The simplest version of the model we consider has a two point distribution, $x \in \{x_1, x_2\}$, with $x_2 > x_1$. In this case, one can define equilibrium in such a way that there is only a small number of different types of possible equilibria, and it is relatively easy to completely characterize the equilibrium set and describe the parameter values that lead to multiplicity. A simple (but not the only) example of multiplicity is for that some parameters individuals could be "faithful" or "unfaithful" in x_1 matches, depending on what they believe others are doing. Perhaps surprisingly, we show that for some parameters there can even be equilibria where agents are "unfaithful" in x_2 matches but "faithful" in x_1 matches, even though $x_2 > x_1$, simply due to endogenous instability.

Moreover, whether there are mulitple equilibria or not, we show that there tends to be inefficiently high levels of search while matched, and this leads to inefficiently high levels of turnover, unemployment, and inequality in equilibrium. When there are multiple equilibria (for some parameters) excess search while matched can be interpreted as a coordination failure: you will search in certain matches because you think others will, even though a better equilibrium exists where agents stop searching in these matches. However, we want to emphasize that there are parameters for which a unique equilibrium exists, which also tends to have inefficiently high levels of turnover, unemployment, and inequality. This inefficiency is more along the lines of a prisoner's dilemna than a coordination failure: you search in certain matches regardless of what others are doing, even though everyone agrees ex ante that this is not a good outcome.

With any finite distribution $x \in \{x_1, x_2 \ldots, x_n\}$, we could define strategies in terms of whether to enter into an x_j match and whether to continue searching while in an x_j match for $j = 1, 2 \ldots, n$, and hence there is a finite number of possible outcomes. In the case where x is drawn from a general distribution, this is not so. We consider equilibria of the following class (which is very natural, although it actually does not include all possible equilibria): agents choose a reservation match value R such that they enter relationships iff $x \geq R$, and a critical match value Q such that they search while matched iff $x \leq Q$. We show that any value of Q in some nondegenerate interval satisfies the steady state equilibrium conditions; i.e., there is a continuum of steady state equilibria. The probability of searching while matched is increasing in Q, and even the best equilibrium tends to have Q above the efficient outcome. Hence, we again find excessive turnover, unemployment, and inequality in equilibrium.

In general, we conclude that endogenous instability is a natural but previously neglected force that can lead to interesting multiplicities and inefficiencies. The remainder of the paper involves making things precise and proving our assertions, some of which may not be at all obvious without the model and the analysis. Section 2 presents the basic notation and assumptions. Section 3 analyzes the simple version with a two-point distribution for x. Section 4 discusses the robustness of the findings to changes in some basic assumptions. Section 5 analyzes the case of a general x distribution. For each version of the model we provide four main results: on the existence and number of equilibria; on the steady state distribution of match quality; on the solution to the social planner's problem; and on the comparison between the efficient and the equilibrium outcomes. Section 5 provides a brief summary and conclusion. Proofs of most results are relegated to the Appendix.[1]

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[1] In terms of related literature, searching while matched (or on-the-job search) is discussed in many places, including Burdett (1977), Mortensen (1978,1988), Pissarides (1994), Webb (1998,1999), Burdett and Mortensen (1998), and Burdett and Coles (1999); see Mortensen and Pissarides (1998) for other references. Those papers do not focus on the issues we study, however, including endogenous instability and the implications for inequality and welfare. Byeonju (1999) argues as we do that there is too much unemployment due to on-the-job search, although his model and results are quite different (e.g., he claims a unique equilibrium). There is a related body of work that studys partnership formation and cooperation, including Kranton (1996), Ghosh and Ray (1996), Ramey and Watson (1997), and Eeckhout (2000). These models and most of the questions asked in this literature are very different from what we do, however, and they are not at all concerned with the types of multiplcities and inefficiencies that we emphasize here.

2 The Basic Framework

There is a $[0,1]$ continuum of infinitely-lived agents who are interested in forming bilateral relationships. While unmatched, they search and receive instantaneous utility b from being single net of any search costs. While searching they meet other agents according to a Poisson process with arrival rate α. Agents are homogeneous ex ante, but matches are heterogeneous ex post: when a pair meet they draw a random variable x giving the instantaneous utility that each would receive if they form a partnership.[2] The distribution of potential match quality is $F(x)$, which is exogenous. The distribution of match quality across existing relationships is $G(x)$, which is endogenous because agents may accept some values of x and reject others.

If you search while matched, you pay cost $d > 0$ and continue to meet new agents at rate α; if you do not search you pay no cost and meet no one new. If you are matched and meet someone new, we make the following base assumption: you must first leave your current partner before you draw the value of x associated with the new person, and you may then form a new relationship, or reject the new person and become unmatched, but you cannot go back to your old partner. This may or may not be realistic, depending on the application, but there are several reasons why it is a good assumption for our purposes. First, the alternative model, where you are allowed to go back, is much more complicated for a general distribution F.

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[2] One can assume that each meeting generates a random surplus X, and agents bargain in such a way that each gets $x = X/2$ (as would follow from standard bargaining theories). Alternatively, one can assume nontrannsferrable utility. Most of what we do does not depend on whether utility is trasferrable; see Section 4, however.

Even for a simple specification of F, we will show the results are still less straightforward to derive the alternative model, and it turns out that the results are qualitatively exactly the same as under the base assumption.

Moreover, our base assumption is appropriate if one wants to focus on the strategic interactions *within* relationships. To explain this, consider the opportunities available when you search. First, your arrival rate of meetings α could depend on the number of agents searching, if the matching technology has non-constant returns. As this has been studied extensively in the past, we assume constant returns. However, even given a constant α, if we allow a matched agent who meets someone new to choose between the new person and his current partner, your *effective* arrival rate will still be endogenous. To see this, note that if matched agents are allowed to choose, they will only enter into a relationship with you if you beat their current value of x, and to know your chances of this you need to know the endogenous distribution G. Under our assumptions you do not need to know G: everyone you meet is effectively unmatched and generates a random draw from F. This allows us to solve the model recursively (see below) and keeps it tractable.

Also, this allows agents to take effective arrival rates as given, and therefore to concentrate on what is happening within relationships when deciding whether to search while matched. A critical aspect of what is happening in your relationship is the search behavior of your partner, who could meet someone else and leave.[3] There are also exogenous teminations at rate σ, *in*

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[3] Given someone is searching while matched, when they meet someone new there is no additional decision to leave or stay with their current partner (i.e., to leave or stay *without* observing the new x; once they observe the new x they are not allowed to stay). This is

addition to the endogenous teminations that occur when your partner leaves, which guarantees that there is always a positive fraction of the population unemployed (i.e., single) in steady state. All of these teminations could be considered involuntarity. You can also voluntarily move from one match directly to another, and you can move from a match to unemployment when meet someone new but find them worse than being single. Hence, although we do not dwell on this interpretation here, the model does have well-defined notions of quits and layoff.[4]

3 A Simple Model

In this section we suppose that $x = x_2$ with probability π and $x = x_1 < x_2$ with probability $1 - \pi$. Agents can be in one of three states: unmatched, in an x_1 relationship, or in an x_2 relationship. The fraction in each state is denoted N_0, N_1, and N_2, where $N_0 + N_1 + N_2 = 1$. Let the payoff, or value, function of an agent in each state be V_0, V_1, and V_2. Agents need to choose strategies for deciding when to accept a match and when to search while matched. Let A_j be the probability that a representative agent agrees to enter into an x_j match, and let S_j be the probability that he searches while in an x_j match, $j = 1, 2$. Sometimes we will be more explicit by saying

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because all new people look identical ex ante, so if an agent would not leave with one new person they would not leave with any, and therefore would not be engaged in costly search in the first place.

[4] This is not true in many models (e.g., the Mortensen - Pissarises model), where all separations are by mutual consent. Later we discuss contracts that could include severance payments, or prenuptial arrangements, which can change the voluntary versus involuntary nature of separations. In any case, nothing much in this paper hinges on one's interpretation of separations as or voluntary versus involuntary, or as quits versus layoffs.

that you choose a_j and s_j taking as given that others choose A_j and S_j; in equilibrium, of course, $a_j = A_j$ and $s_j = S_j$. We focus for now on pure strategy equilibria (mixed strategies are considered later).

The value functions satisfy the standard continuous time dynamic programming equations,

$$\begin{aligned}
rV_0 &= b + \alpha\pi A_2(V_2 - V_0) + \alpha(1-\pi)A_1(V_1 - V_0) \\
rV_1 &= x_1 + (\sigma + S_1\alpha)(V_0 - V_1) + s_1\Sigma_1 \qquad (1) \\
rV_2 &= x_2 + (\sigma + S_2\alpha)(V_0 - V_2) + s_2\Sigma_2,
\end{aligned}$$

where Σ_i is the net gain from searching while in an x_i match,

$$\begin{aligned}
\Sigma_1 &= \alpha\pi[A_2V_2 + (1-A_2)V_0 - V_1] + \alpha(1-\pi)(1-A_1)(V_0 - V_1) - d \\
\Sigma_2 &= \alpha\pi(1-A_2)(V_0 - V_2) + \alpha(1-\pi)[A_1V_1 + (1-A_1)V_0 - V_2] - d.
\end{aligned}$$

For example, the third equation in (1) equates the flow value rV_2 to the sum of three terms. The first is the instantaneous match utility, x_2. The second is the probability you become unmatched, either because of an exogenous separation or because your partner meets someone new, $\sigma + S_2\alpha$, times the capital loss, $V_0 - V_2$. The final terms is your search decision s_2 times the net gain Σ_2.[5]

Given (A_1, A_2, S_1, S_2), we show below that there is a unique steady state (N_0, N_1, N_2). We emphasize, however, that we do not need to know the steady state to analyze search behavior, since it does not enter (1) (i.e., the model

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[5] One could be more explicit in terms of a_j and A_j, but we have written the equations in the text using the fact that, given x, the two parties always agree on whether to form a relationship.

is recursive). A *steady state equilibrium* is a list including value functions (V_0, V_1, V_2) satisfying (1), the steady state (N_0, N_1, N_2) satisfying conditions to be given below, and strategies (A_1, A_2, S_1, S_2) satisfying the following best response conditions,

$$A_j = \begin{cases} 1 & \text{if } \Delta_j > 0 \\ [0,1] & \text{if } \Delta_j = 0 \\ 0 & \text{if } \Delta_j < 0 \end{cases}, \qquad S_j = \begin{cases} 1 & \text{if } \Sigma_j > 0 \\ [0,1] & \text{if } \Sigma_j = 0 \\ 0 & \text{if } \Sigma_j < 0 \end{cases} \tag{2}$$

where $\Delta_j = V_1 - V_0$ is the net gain from accepting an x_j match, and Σ_j is the net gain from searching while in an x_i match defined above.

As we will show, of all possible pure strategy profiles, only five potentially constitute equilibria. First, there is a "degenerate" or *type D* equilibrium where agents reject all matches: $A_1 = A_2 = 0$ (and S_i turns out to be irrelevant). Second, there is what we call a "choosy" or *type C* equilibrium where agents accept x_2 but reject x_1 and, since they would not accept x_1, they do not search in x_2 matches: $A_1 = 0$, $A_2 = 1$ and $S_2 = 0$. Third, there is a "faithful" or *type F* equilibrium where agents accept x_1 as well as x_2 and do not search in either case: $A_1 = A_2 = 1$ and $S_1 = S_2 = 0$. Fourth, there is an "unfaithful" or *type U* equilibrium, in which agents accept both matches and continue to search in x_1 matches: $A_1 = A_2 = 1$, $S_1 = 1$ and $S_2 = 0$. Finally, there is a "perverse" or *type P* equilibrium where agents accept both and, perhaps counter to intuition, search in x_2 but not x_1 matches: $A_1 = A_2 = 1$, $S_1 = 0$ and $S_2 = 1$.

The following proposition describes exactly when each of the different types of equilibria exist. The proof, which is involves checking every possible stategy profile, is given in the Appendix in full detail (although the method should be clear from one or two cases).

Proposition 1 *There are five potential types of equilibria:*

$$\begin{array}{lll} \textit{type } D: & A_1 = A_2 = 0 & \text{exists iff } x_2 \leq b \\ \textit{type } F: & A_1 = A_2 = 1, S_1 = S_2 = 0 & \text{exists iff } x_2 \leq y_1, y_2 \\ \textit{type } U: & A_1 = A_2 = 1, S_1 = 1, S_2 = 0 & \text{exists iff } x_1 \leq b + d,\ x_2 \geq y_3 \\ \textit{type } P: & A_1 = A_2 = 1, S_1 = 0, S_2 = 1 & \text{exists iff } b + d \leq x_2 \leq y_4 \\ \textit{type } C: & A_1 = 0, A_2 = 1, S_2 = 0 & \text{exists iff } x_1 \leq b + d,\ x_2 \geq y_1 \end{array}$$

where the critical values y_j are given by

$$\begin{aligned} y_1 &= \frac{r+\sigma+\alpha\pi}{\alpha\pi}x_1 - \frac{r+\sigma}{\alpha\pi}b \\ y_2 &= x_1 + \frac{r+\sigma}{\alpha\pi}d \\ y_3 &= \frac{r+\sigma+\alpha}{r+\sigma+2\alpha}x_1 + \frac{\alpha}{r+\sigma+2\alpha}b + \frac{(r+\sigma)(r+\sigma+2\alpha)+\alpha^2\pi}{\alpha\pi\,(r+\sigma+2\alpha)}d \\ y_4 &= \frac{r+\sigma+2\alpha}{r+\sigma+\alpha}x_1 - \frac{\alpha}{r+\sigma+\alpha}b - \frac{(r+\sigma)(r+\sigma+2\alpha)+\alpha^2(1-\pi)}{\alpha(1-\pi)(r+\sigma+\alpha)}d. \end{aligned}$$

There are no other pure strategy equilibria.

The regions where the different equilibria exist are depicted in (x_1, x_2) space in Figure 1 (*type D* equilibrium exists in the region labeled D, and so on). Notice, for example, that the *type F* equilibrium exists when x_1 is big enough that agents accept it, and x_2 is small enough that agents stop searching once they do accept x_1. For the *type U* equilibrium, x_2 must be large enough to make agents search in x_1 matches, but we also need x_1 large enough that agents prefer to accept x_1 and search rather that reject it and search while unmatched. Observe that when $x_2 \in (y_3, y_2)$ the *type F* and *type U* equilibria coexist; hence agents in x_1 matches may either be "faithful" or "unfaithful" depending on what other agents are doing. This is endogenous instability.[6]

Endogenous instability is sufficiently powerful that for some parameters there exists the "perverse" equilibrium where $V_1 > V_2$, even though $x_1 < x_2$, simply because agents believe that x_2 matches will be unstable and these matches are in fact unstable because people in x_2 matches search. This is only possible if x_1 is not too much less than x_2, however, since agents will only sacrifice so much utility for security, and if x_1 and x_2 are large relative to $b + d$, since it is a high cost of separation that makes security important. Also notice that whenever the *type P* equilibrium exists there coexists another equilibrium: the always coexists it's mirror image, the *type U* equilibrium, and sometimes there coexists both the *type U* and *type F* equilibria. In any case, we do not intend to dwell on "perverse" outcomes, but mention the *type P* equilibrium as an extreme example of endogenous instability.[7]

We next proceed to consider efficiency, defined in terms of a standard social planner's welfare criterion:

$$W = N_0 V_0 + N_1 V_1 + N_2 V_2. \tag{3}$$

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[6] When *type F* and *type U* equilibria coexist, one can also construct a mixed-strategy equilibrium where agents in x_1 matches search with probability $S_1 \in (0, 1)$. We omit the routine calculations in the interests of space, but details are available on request.

[7] A special case of a *type P* equilibrium occurs when $x_1 = x_2$, which means that there is no fundamental difference between matches but people simply believe that certain relationships will be unstable. This has an interpretation in terms of *discrimination*. For example, suppose that agents are distinguished by one of two identifiable but otherwise irrelevant characteristics, say black and white. It is possible for individuals to believe that black-white relationships will be less stable and hence less desirable than black-black or white-white relationships, and for this belief to be true in equilibrium, even if color has no intrinsic impact on payoffs.

As $x_2 < b$ implies the efficient outcome is obviously $A_1 = A_2 = 0$, let us proceed to the more interesting case where $x_2 > b$. Then any efficient outcome clearly entails $A_2 = 1$ and $S_2 = 0$. Hence, the *type P* equilibrium cannot possibly be efficient, so we ignore it for now and ask about the efficiency properties of other equilibria. That is, we ask when the planner prefers the *type C*, *type F* or *type U* strategies. The answer is given below, where again we relegate the proof to the Appendix.

Proposition 2 *Given $x_2 > b$, the strategies that maximize W are as follows:*

$$\begin{array}{lll} \textit{type } C: & A_1 = 0 & \text{if } x_1 \leq b + d \text{and } x_2 \geq y_A \\ \textit{type } F: & A_x = 1, S_1 = 0 & \text{if } x_2 \leq y_A \text{and } x_2 \leq y_S \\ \textit{type } U: & A_x = 1, S_1 = 1 & \text{if } x_1 \geq b + d \text{and } x_2 \geq y_S, \end{array}$$

where the critical values y_j are given by

$$\begin{aligned} y_A &= \frac{\alpha\pi + \sigma}{\alpha\pi} x_1 - \frac{1}{\alpha\pi} b \\ y_S &= \frac{\pi(\sigma + 2\alpha) + \sigma}{\pi(\sigma + 2\alpha)} x_1 - \frac{\sigma}{\pi(\sigma + 2\alpha)} b + \frac{\sigma(\sigma + \alpha)}{\alpha\pi(\sigma + 2\alpha)} d. \end{aligned}$$

To compare equilibria and efficient outcomes, it is best to begin with the limiting case $r \to 0$, since then the y_1 defined in Proposition 1 coincides with the y_A defined in Proposition 2. This means that the region where $A_1 = 1$ is an equilibrium coincides with the region where it is efficient, and given that A_1 is efficient, we can concentrate on the efficiency of S_1. Figure 2 shows a version of Figure 1 drawn with $r = 0$, and highlights two regions in which the equilibrium differs from the planner's solution (in all other regions, equilibrium is efficient). In the region labeled 1, the planner chooses the *type F* strategy but the unique equilibrium is *type U*. Here, there is unambiguously too much search, in the sense that $S_1 = 1$ is the unique equilibrium but the efficient solution is $S_1 = 0$. In the region labeled 2, the planner again chooses the *type F* strategy, but now there are multiple equilibria, including *type F* but also including *type U*. In this case, we may or may not have too much search.

The conclusion is that, at least when r is not too big, there is a tendency towards too much search. Heuristically, the reason is that if your partner decides to search while matched, they take into account their own costs and benefits but neglect the cost on you.[8] Notice that the inefficiency takes the form of a coordination failure in region 2, where there are both good and bad equilibria: you will search in an x_1 match, counter to social efficiency, if and only if everyone else does. In contrast, in region 1 the inefficiency takes the form of a prisoner's dilemma: you will search in an x_1 match, counter to social efficiency, regardless of what everyone else does.

Before discussing the welfare results for large r, we first want to compare the distributions of agents across states, in the equilibrium and the efficient outcomes. Begin by writing the net flow of agents into x_1 matches as

$$\dot{N}_1 = N_0\alpha(1-\pi)A_1 + N_2 S_2 \alpha(1-\pi)A_1 - N_1\psi_1, \tag{4}$$

where $\psi_1 = \sigma + S_1\alpha + S_1\alpha(1-\pi)(1-A_1) + S_1\alpha\pi$. The first term represents

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[8] One may think there is an additional effect, which is that when other agents search they might meet your partner, who will then abandon you. While this could be true in some models, it is not true here due to our assumptions on the matching technology: when other agents search, they increase your arrival rate since they might meet you, but also decrease it since they might meet others who could have met you, and these effects exactly cancel under constant returns. Although this is not uninteresting, we stick to constant returns for the reasons discussed above.

single agents who find an x_1 match and accept. The second term represents agents in x_2 matches who search, find an x_1 match and accept. The final term is flow out of x_1 matches, including agents whose relationships break up exogenously, those who are abandoned by their partners, those who are searching and get an x_1 draw and reject it, and those who are searching and get an x_2 draw whether they accept or not. There is a symmetric expression for $\dot{N}_2$, and steady state solves $\dot{N}_1 = \dot{N}_2 = 0$. Letting superscripts indicate the steady state in a particular equilibrium (e.g., N_0^C is the number of unmatched agents in *type C* equilibrium), we have the following:

Proposition 3 *There is a unique steady corresponding to each type of equilibrium, given by*

$$\begin{array}{lll}
N_0^D = 1 & N_1^D = 0 & N_2^D = 0 \\
N_0^C = \dfrac{\sigma}{\alpha\pi + \sigma} & N_1^C = 0 & N_2^C = \dfrac{\alpha\pi}{\alpha\pi + \sigma} \\
N_0^F = \dfrac{\sigma}{\alpha + \sigma} & N_1^F = \dfrac{\alpha(1-\pi)}{\alpha + \sigma} & N_2^F = \dfrac{\alpha\pi}{\alpha + \sigma} \\
N_0^U = \dfrac{\sigma(\sigma + \alpha + \alpha\pi)}{\kappa_U} & N_1^U = \dfrac{\sigma\alpha(1-\pi)}{\kappa_U} & N_2^U = \dfrac{\alpha\pi(\sigma + 2\alpha)}{\kappa_U} \\
N_0^P = \dfrac{\sigma(\sigma + 2\alpha - \alpha\pi)}{(\sigma + 2\alpha)\kappa_P} & N_1^P = \dfrac{\alpha(1-\pi)}{\kappa_P} & N_2^P = \dfrac{\sigma\alpha\pi}{(\sigma + 2\alpha)\kappa_P}
\end{array}$$

where $\kappa_U = (\sigma + 2\alpha)(\alpha\pi + \sigma)$ *and* $\kappa_P = \alpha(1-\pi) + \sigma$*.*

These results all following from routine algebra, so we omit any proof. The key observation is that $N_0^U > N_0^F$, $N_2^U > N_2^F$ and $N_1^U < N_1^F$. The is relevant because when the equilibrium and efficient outcomes differ, it is because we are in a *type U* equilibrium but the planner's preferred strategies are *type F*. This implies that the planner prefers fewer unmatched agents, fewer x_2 matches, and more x_1 matches than obtain in equilibrium. In other words, the efficient outcome involves less unemployment and less inequality. Intuitively, too much search comes with too much inequality, simply because what agents are searching for is to move up in the distribution, and also too much unemployment, because one way they move up in the distribution is by abandoning their current partners.[9]

The welfare results stated above (although not the steady state results) are for the case $r \approx 0$. The effect that generates too much search is always there for $r > 0$, but for big r there is another effect that goes the other way. This other effect implies too little search, according to W, because impatient agents are less inclined than the planner to reject an x_1 match and to search in an x_1 match since the private gains from doing so accrue only in the future but the cost is paid today. Figure 3 provides a version of Figure 2 with $r > 0$, and shows that, in addition to the regions 1 and 2 with too much search, there are three new regions with too little search. In the region labeled 3, the planner chooses the *type C* strategy but the unique equilibrium is *type F*; in the region labeled 4, the planner chooses the *type U* strategy but the unique equilibrium is *type F*; and in the region labeled 5, the planner chooses the *type U* strategy but *type U* and *type F* equilibria exist.

The effect that may lead to too little search, due exclusively to differences in discount rates between the planner and private agents, is well understood

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[9] For the record, *type P* equilibrium has too many unmatched agents and too many x_1 matches compared to the efficient outcome.

and not especially interesting. What is novel here is the effect that tends to generate too much search, which in any event always dominates for small r. We summarize what has been shown above in terms of welfare with the following statement.

Proposition 4 *At least for r not too big, either the equilibrium is efficient, or the equilibrium has $S_1 = 1$ while the efficient outcome has $S_1 = 0$, which means the equilibrium has too much search, too much unemployment, and too much inequality.*

4 Robustness

In this section we discuss some robustness issues. To begin, we sketch the effect of changing the base assumption that you cannot stay with your old partner when you meet someone new. If we change this assumption, then in principle we need to generalize acceptance strategies by letting A_j^i be the probability that an agent enters into an x_j match *conditional* on currently being in state $i = 0, 1, 2$. In the base model we did not need to condition acceptance decisions on the agent's state, since everyone is effectively single when they decide whether to accept x_j (becasue they are not allowed to stay with their current partner). This would complicate things, in general; however, given a two-point distribution for x, one can show that $A_j^i = A_j$.[10]

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[10] First note that we have the same five types of equilibria with the alternative as we had under the base assumption. Now, clearly, in a *type D* equilibrium we have $A_j^i = 0$ for all i, j. In a *type C* equilibrium, $A_1^0 = 0$ and $A_2^0 = 1$, since $V_2 > V_0 > V_1$. This implies $A_1^2 = 0$ and $A_2^1 = 1$. So we have $A_1^i = 0$ and $A_2^i = 1$ in this case. In a *type F*, *type U* or *type P* equilibrium, $A_1^0 = A_2^0 = 1$. If $A_2^1 = 1$ then $A_2^0 = 1$ and if $A_2^1 = 0$, then $S_1 = 0$ and

Hence, we do not actually have to worry about conditioning acceptance decisions on the state. Given this, the analogue of (1) is

$$\begin{aligned}
rV_0 &= b + \alpha(q_0 + q_1)\pi A_2(V_2 - V_0) + \alpha(q_0 + q_2)(1-\pi)A_1(V_1 - V_0) \\
rV_1 &= x_1 + [\sigma + S_1\alpha(q_0 + q_1)\pi A_2](V_0 - V_1) + S_1\Sigma_1 \qquad (5) \\
rV_2 &= x_2 + [\sigma + S_2\alpha(q_0 + q_2)(1-\pi)A_1](V_0 - V_2) + S_2\Sigma_2,
\end{aligned}$$

where

$$\begin{aligned}
\Sigma_1 &= \alpha(q_0 + q_1)\pi A_2(V_2 - V_1) - d \\
\Sigma_2 &= \alpha(q_0 + q_2)(1-\pi)A_1(V_1 - V_2) - d.
\end{aligned}$$

In these expressions q_0 and q_i are the probabilities that you meet an unmatched agent and an agent in a x_i match, respectively:

$$q_0 = \frac{N_0}{N_0 + S_1N_1 + S_2N_2} \text{ and } q_i = \frac{S_iN_i}{N_0 + S_1N_1 + S_2N_2},\ i = 1, 2.$$

This model is not recursive, because the N_i's appear in (5), so we need to solve for steady state before characterizing equilibria.[11]

It turns out that, although messier, the results are qualitatively the same in the alternative model as those shown in Figure 1. Indeed, the *type D*, *type*

A_2^1 is irrelevant. For A_1^2, we have a similar result. So again we can set $A_j^i = A_j$ in all the relevant cases. The establishes the claim.

[11] For the record, in *type D*, *type C* or *type F* equilibria the steady state is actually the same as in the base model. In the *type U* and *type P* equilibria, we have

$$\begin{aligned}
&N_0^U = \tfrac{\sigma}{\alpha\pi+\sigma}\left[1 - \tfrac{2\alpha+\sigma-\xi_U}{2\alpha(1-\pi)}\right] \quad N_1^U = \tfrac{\sigma(2\alpha+\sigma-\xi_U)}{(\alpha\pi+\sigma)[2\alpha(1-\pi)]} \quad N_2^U = \tfrac{\alpha\pi}{\alpha\pi+\sigma} \\
&N_0^P = \tfrac{\sigma}{\alpha(1-\pi)+\sigma}\left[1 - \tfrac{2\alpha+\sigma-\xi_P}{2\alpha\pi}\right] \quad N_1^P = \tfrac{\alpha(1-\pi)}{\alpha(1-\pi)+\sigma} \quad N_2^P = \tfrac{\sigma(2\alpha+\sigma-\xi_P)}{2\alpha\pi[\alpha(1-\pi)+\sigma]}
\end{aligned}$$

where $\xi_U = \sqrt{(\sigma + 2\alpha\pi)(\sigma + 4\alpha - 2\alpha\pi)}$ and $\xi_P = \sqrt{(\sigma + 2\alpha + 2\alpha\pi)(\sigma + 2\alpha - 2\alpha\pi)}$. Substituting the N_j's into the Bellman equations yields a complicated but not intractable system (details are available upon request).

C and *type* F equilibria exist in exactly the same regions of parameter space, while the *type* U and *type* P equilibria exist for smaller regions than in the base model. It may be surprising that the *type* U and *type* P equilibria exist for smaller regions, since it would seem that our base assumption discourages on-the-job search. However, with a two point x distribution it really does not. Suppose, e.g., you are in an x_1 match and searching. Since next person you meet cannot generate any less than x_1, the base assumption that you cannot stay with your old partner is not binding – it simply serves as a tie-breaking rule. However, there is a general equilibrium effect: if the tie-breaking rule is to go with the new person, relationships are less stable, and agents are therefore more inclined to search.

We conclude several things about the alternative model. First, it is more complicated than the base model because it is not recursive. Second, at least for the case where x has a two-point distribution, the alternative model generates the same qualitative implications as the base model: the exact regions where some of the equilibria exist differ slightly, but the basic theorems on existence, multiplicity, and efficiency continue to hold (in particular, since W does not depend on which tie-breaking rule, we still have a version of Figure 2 showing that equilibrium is either efficient or has too much search). Finally, note that for the case of a general x distribution, the model without our base assumption is very much more complicated. These considerations all suggest that when we study the case of a general distribution for x in the next section, it makes sense to go with the base assumption.

Before we move to the general model, we mention one more robustness issue.

5 The General Model

The general match quality distribution is $F(x)$. If it is differentiable we denote the density by $f(x)$, but we do not especially need differentiability. We will focus on equilibria with the following property: unmatched agents accept partners iff $x \geq R$ where R is called the reservation match quality; and matched agents search iff $x \leq Q$ where Q is called the critical match quality.[12] Moreover, we concentrate on equilibria with $Q > R$, since we want some on-the-job search (sufficient conditions for this are given below). An equilibrium now is defined as a list including the value functions for unmatched agents and agents in relationships with match value x, $[V_0, V(x)]$, a steady state described by the number of unmatched agents and the distribution of match values across existing relationships, $[N_0, G(x)]$, and strategies (R, Q), satisfying conditions to be given below.

To begin the analysis, observe that V_0 satisfies the standard equation from search theory

$$rV_0 = b + \alpha \int_R^\infty [V(z) - V_0] dF(z), \tag{6}$$

where R is the reservation match quality, satisfying $V(R) = V_0$. Similarly, $V(x)$ satisfies

$$rV(x) = x + (\sigma + S\alpha)[V_0 - V(x)] + s\Sigma(x) \tag{7}$$

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[12] That there in general may be equilibria not in this class can be seen from the previous section, where in a *type P* equilibrium agents search in x_2 but not x_1 matches. Our point will be to show that there are lots of equilibria, even within this class.

where

$$\Sigma(x) = \alpha[V_0 - V(x)] + \alpha \int_R^\infty [V(z) - V_0] dF(z) - d.$$

In any equilibrium with $Q > R$, agents search in relationships at the reservation match value (i.e., $x = R$ implies $s = 1$). Therefore,

$$rV(R) = R + \alpha \int_R^\infty [V(z) - V_0] dF(z) - d. \tag{8}$$

Comparing (8) and (6), one sees immediately that $R = b + d$, and so unmatched agents form relationships iff the instantaneous return net of search cost, $x - d$, exceeds b.[13]

We next derive a quasi-reduced form version of $V(x)$. First rewrite (7) as

$$rV(x) = x - sd + (\sigma + S\alpha + s\alpha)[V_0 - V(x)] + s\alpha I,$$

where $I = \int_{b+d}^\infty [V(z) - V_0] dF(z)$ does not depend on x. Then insert $V_0 = (b + \alpha I)/r$ and rearrange to yield

$$V(x) = \frac{r(x - sd) + (\sigma + S\alpha + s\alpha)b + (\sigma + S\alpha + s\alpha + sr)\alpha I}{r(r + \sigma + S\alpha + s\alpha)}. \tag{9}$$

For future reference, note that

$$I = \int_{b+d}^\infty [1 - F(z)]V'(z)dz = \int_{b+d}^Q \frac{[1 - F(z)]dz}{r + \sigma + 2\alpha} + \int_Q^\infty \frac{[1 - F(z)]dz}{r + \sigma} dz \tag{10}$$

after integrating by parts and inserting $V'(x)$ from (9). This gives $I = I(Q)$ as a function of Q; otherwise, I depends only on exogenous variables.

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[13] The result $R = b + d$ is only true in general in an equilibrium with $Q > R$. Recall from the previous section that there could exist a *type F* equilibrium where agents accept matches with $x < b + d$, which means $R < b + d$; however, in this equilibrium agents do not search in these matches, so $Q < R$, contrary to the maintained assumption in this section.

Equation (9) explicitly expresses a matched agent's payoff as a function of match quality x, his search behavior s, the search behavior of his partner S, and the equilibrium value of Q, which determines the behavior of other agents since it is taken as given that they search while matched iff $x \leq Q$. Denoting this by $v_{sS}(x, Q)$, we have:

$$\begin{aligned} v_{11}(x,Q) &= \frac{r(x-d)+(\sigma+2\alpha)b+(\sigma+2\alpha+r)\alpha I(Q)}{r(r+\sigma+2\alpha)} \\ v_{01}(x,Q) &= \frac{rx+(\sigma+\alpha)b+(\sigma+\alpha)\alpha I(Q)}{r(r+\sigma+\alpha)} \\ v_{10}(x,Q) &= \frac{r(x-d)+(\sigma+\alpha)b+(\sigma+\alpha+r)\alpha I(Q)}{r(r+\sigma+\alpha)} \\ v_{00}(x,Q) &= \frac{rx+\sigma b+\sigma\alpha I(Q)}{r(r+\sigma)}. \end{aligned}$$

Figure 4 shows the four v_{ij} functions. Notice that they are linear in x, with $\partial v_{00}/\partial x > \partial v_{01}/\partial x = \partial v_{10}/\partial x > \partial v_{11}/\partial x$. Also, as shown in the figure, we have $v_{10} = v_{11} > v_{01}, v_{00}$ at $x = b + d$, which follows from $Q > R = b + d$. Given all agents search iff $x \leq Q$, your value function $V(x)$ is given by $\max\{v_{01}, v_{11}\}$ when your partner is searching and $\max\{v_{00}, v_{10}\}$ when your partner is not searching, as shown by the thick line in the diagram.

Given any $Q > b + d$, it should be clear from Figure 4 that there is a unique $q_0(Q) > b + d$ such that $v_{00} = v_{10}$; that is, given your partner is not searching, your best response switches from searching to not searching when x crosses $q_0(Q)$. Similarly, there is a unique $q_1(Q) > b + d$ such that $v_{01} = v_{11}$; that is, given that your partner is searching, your best response switches from searching to not searching as x crosses $q_1(Q)$. Figure 4 depicts the situation with $b + d < q_0(Q) < Q < q_1(Q)$. We claim that this implies it is an equilibrium for all agents to search while matched iff $x \leq Q$. To see why, suppose that everyone else searches iff $x \leq Q$: then if you are in a match with $x \leq Q$, your partner searches, and so you want to search because $x < q_1(Q)$; and if you are in a match with $x > Q$, your partner does not search, and so you do not because $x > q_0(Q)$. This establishes the claim.

It may not be obvious that the situation depicted in the figure, $b + d < q_0(Q) < Q < q_1(Q)$, will actually arise. To show that it does, we first equate $v_{0j} = v_{1j}$ and rearrange to yield

$$\begin{aligned} \alpha q_0(Q) &= \alpha b - (\sigma + r)d + (\sigma + \alpha + r)\alpha I(Q) \\ \alpha q_1(Q) &= \alpha b - (\sigma + \alpha + r)d + (\sigma + 2\alpha + r)\alpha I(Q). \end{aligned}$$

As seen in Figure 5, the functions $q_0(Q)$ and $q_1(Q)$ are decreasing, because $I(Q)$ is. Therefore each has at most one fixed point. We now make the following mild parameter restriction, which always holds if d is not too big:

$$\frac{\alpha}{r + \sigma} \int_{b+d}^{\infty} [1 - F(z)] dz > d. \tag{11}$$

This basically says that F is such there is some net gain to searching if employed at $b + d$ (if this was not true then there would never be on-the-job search). It is not hard to verify that (11) guarantees $q_0(b + d) > b + d$, and so the fixed point of $q_0(Q)$, call it $\underline{q}$, satisfies $\underline{q} > b + d$. Also, $\underline{q} > b + d$ guarantees $q_1(\underline{q}) > \underline{q}$, and so the fixed point of $q_1(Q)$, call it $\overline{q}$, satisfies $\overline{q} > \underline{q}$.

Summarizing the above analysis, we have the following.

Proposition 5 *Assuming (11) holds, we have* $b + d < \underline{q} < \overline{q}$, *as depicted in Figure 5. Then we can choose any* $Q \in [\underline{q}, \overline{q}]$ *and have* $b + d < q_0(Q) \leq Q \leq$

$q_1(Q)$*, as depicted in Figure 4. This means that any* $Q \in [\underline{q}, \overline{q}]$ *is consistent with equilibrium.*

What is behind this continuum of equilibria? One answer goes along the following lines. Consider a simple two-player game where agents can either cooperate or defect, parameterized by ρ, representing, say, the payoff to joint cooperation. Suppose that the game has multiple equilibria, one where both agents cooperate and another where they both defect, for all ρ in some interval (ρ_1, ρ_2). Then imagine agents choosing strategies before ρ is observed of the following form: cooperate if $\rho \geq \hat{\rho}$ and defect if $\rho < \hat{\rho}$. A symmetric equilibrium obtains if they choose the same $\hat{\rho}$. It should be clear that there can be a continuum of such equilibria: for any $\hat{\rho} \in (\rho_1, \rho_2)$, this strategy of cooperating iff $\rho \geq \hat{\rho}$ constitutes a best response to itself (see Frankel, Morris and Pauzer [1999], including references to earlier analyses of similar games).

Intuitively, whatever fundamental impact ρ may have in the above game, it can also be used as a signal or coordinating device determining which equilibrium to play once ρ is realized. Something very similar is going on in our model. The realization of x has fundamental value as the utility you get from a match, but it can also be a signal indicating which equilibrium is to be played in that match. Since we know from the analysis of the simple model in the previous section that there can be multiple equilibria in the game where partners choose whether to search, using the value of x relative to Q as a coordination device is consistent with equilibrium for any Q in some range.

Of course, if x is too small (big), then agents will (will not) want to search regardless of what their partner is doing; this is why the range of equilibrium Q is bounded.[14]

So far, we know $R = b + d$, and we know that any Q in $[\underline{q}, \overline{q}]$ is consistent with equilibrium. To complete the characterization we need to describe the distribution of agents across states. With a little work, described in the Appendix, we have the following.

Proposition 6 *Given equilibrium Q and $R = b + d$, the steady state unemployment rate is*

$$N_0 = \frac{\sigma[\sigma + 2\alpha - \alpha F(Q) + \alpha F(R)]}{(\sigma + 2\alpha)[\sigma + \alpha - \alpha F(Q)]}, \tag{12}$$

and the distribution of x across existing matches is

$$G(x) = \begin{cases} \dfrac{\sigma[F(x) - F(R)]}{\sigma[1 - F(R)] + 2\alpha - 2\alpha F(Q)} & \text{if } x \in [R, Q] \\[2ex] \dfrac{\sigma[F(x) - F(R)] + 2\alpha[F(x) - F(Q)]}{\sigma[1 - F(R)] + 2\alpha - 2\alpha F(Q)} & \text{if } x > Q. \end{cases} \tag{13}$$

Notice G transforms F by truncating it below R, scaling it between R and Q, and scaling it differently above Q. Also, $G(x) \leq F(x)$ for all x with strict inequality on the interior of the support (stochastic dominance). Also, G is continuous at $x \neq Q$ as long as F is, but in any case G has a kink at Q.

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[14] This interpretation also suggests the result is *not* due to the fact that we assume search is a discrete choice. Even if s is a continuous variable, as long as their are multiple equilibria in the game of deciding how much to search while matched, it should be possible to generate a continuum of equilibria along very similar lines.

Thus, if F has density f then G has density:

$$g(x) = \begin{cases} \dfrac{\sigma f(x)}{\sigma[1-F(R)] + 2\alpha - 2\alpha F(Q)} & \text{if } x \in [R, Q] \\ \\ \dfrac{(\sigma + 2\alpha) f(x)}{\sigma[1-F(R)] + 2\alpha - 2\alpha F(Q)} & \text{if } x > Q \end{cases} \tag{14}$$

Increasing Q from Q_1 to Q_2 shifts g up for $x < Q_1$ and $x > Q_2$, and shifts g down for $x \in (Q_1, Q_2)$; see Figure 6, which is drawn assuming f is log normal. Hence, equilibria with higher Q entail more inequality. Also, from (12), increasing Q raises N_0. Hence, equilibria with higher Q entail more unemployment.

It remains to discuss welfare. In the class of outcomes under consideration, given and R and Q, we have

$$W = N_0 V_0 + (1 - N_0)\left[\int_R^Q v_{11}(x, Q)dG(x) + \int_Q^\infty v_{00}(x, Q)dG(x)\right]. \tag{15}$$

After inserting the endogenous steady state and value functions, after some tedious algebra this can be simplified to

$$W = \frac{\chi(Q, R)b + \alpha\sigma \int_R^Q (x - d)dF(x) + \alpha(\sigma + 2\alpha)\int_Q^\infty x dF(x)}{r(\sigma + 2\alpha)[\sigma + \alpha - \alpha F(Q)]}, \tag{16}$$

where $\chi(Q, R) = \sigma[\sigma + 2\alpha - \alpha F(Q) + \alpha F(R)]$. We now solve the planner's problem of maximizing W by choosing R and Q. To ease the presentation, we assume for here that F has density f. Also, we concentrate on the case $Q > R$ (since otherwise there is no on-the-job search), a sufficient condition for which is[15]

$$\int_{b+d}^\infty [1 - F(x)]dx > \frac{\sigma d}{\alpha}. \tag{17}$$

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[15] In the proof of the proposition we show the solution to the planner's problem entails

In the Appendix we show the following:

Proposition 7 *Assuming $Q > R$, the solution to the planner's problem is $R = b + d$ and the value of Q that solves $T^o(Q) = 0$, where*

$$\begin{aligned} T^o(Q) &= 2\alpha\left[\sigma + \alpha - \alpha F(Q)\right]Q - \alpha\sigma\left[1 + F(R)\right]b + \sigma\left[\sigma + \alpha - \alpha F(R)\right]d \\ &\quad -\alpha\sigma\int_R^Q (x-d)dF(x) - \alpha(\sigma + 2\alpha)\int_Q^\infty xdF(x). \end{aligned} \tag{18}$$

Among other things, the above establishes that the equilibrium $R = b + d$ coincides with the efficient solution.[16] We now show that the equilibrium Q is too high, at least in the limiting case $r \to 0$; this means that, exacty as in the previous section, there tends to be too much search in equilibrium, at least for small r. Again, the proof is in the Appendix.

Proposition 8 *Given $Q > R$, the planner's choice of Q is below $\underline{q}$ and therefore below any equilibrium Q, at least for r not too big.*

Summarizing the results in this section, here is what has been established. There exist values $\underline{q}$ and $\overline{q} > \underline{q}$ such that any $Q \in [\underline{q}, \overline{q}]$ is consistent with equilibrium. Given (11), we have $\underline{q} > b + d$, and therefore $Q > b + d$ in any equilibrium. Assuming $Q > R$, in any equilibrium we have seen that $R = b + d$ is efficient, but at least for r not too big Q is less than the solution

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$R = b + d$ and $Q = Q^o$ where Q^o satisfies $T^o(Q) = 0$. Since T^o is an increasing function, and (17) is merely a simplification of $T^o(b + d) < 0$, it guarantees that $Q^o > b + d = R$. Notice that if we let $r \to 0$ in (11) then it is easy to see that it will hold as long as (17) holds; hence, for small r (which is the interesting case below), the former is not binding.

[16] The efficiency of R is predicated on $R < Q$. We know from the model with $x \in \{x_1, x_2\}$ in the previous section that R can be inefficient, at least if $r > 0$, as in region 3 in Figure 3 (where the planner chooses the *type C* strategy but the equilibrium is *type F*).

to the planner's problem – i.e., all equilibria have too much search. Therefore, given the steady state solution for N_0 and $G(x)$, we know that all equilibria also have too much unemployment and too much inequality.

6 Conclusion

Many extensions of the above analysis are possible. An alternative model worth investigating would have ex ante heterogeneity, as opposed to homogeneous agents and match specific values of x. It would also be worth investigating a model with a continuous choice of search intensity, although, as we indicated above, there is reason to suspect that our main results would survive. Many other generalizations and extensions are possible, of course, but we expect the basic economic results to be robust. A potentially interesting extension would have partners potentially disagree on the match value, since you may need to think about a counteroffer when your partner meets someone new, and you may also want to think about offering your partner enough to keep them off the market in the first place.

Another idea for future work would be to try to assess the empirical relevance of endogenous instability in the context of certain labor markets, like sports or academics, say, and in the context of the marriage market. Is there too much turnover in these markets? Also, is there a chance that the frequently discussed differences between the American and European labor markets have anything to do with endogenous instability? It seems the former does have more turnover and more inequality, but the latter has more unemployment, so endogenous instability cannot be the *only* thing leading to the differences between American and European labor markets. However, a model that combines other institutional factors with the kinds of effects analyzed here may be relevant. We leave this to future work.

7 Appendix

Proof of Proposition 1: To begin, we can easily rule out anything other than the five cases that are listed. For example, suppose $A_1 = 0$, $A_2 = 1$, and $S_2 = 1$. These strategies imply $\Sigma_2 = \alpha(1-\pi)(V_0 - V_2) - dS_1$ and $\Delta_2 = V_2 - V_0$; but the best response conditions require $\Sigma_2 > 0$ and $\Delta_2 > 0$, which is a contradiction. The other cases are similar. The next step is to derive parameter values for which each of the remaining five candidate equilibria exist.

Consider first a *type D* equilibrium, where $A_1 = A_2 = 0$ and $rV_0 = b$. We use the unimprovability principle: to check whether strategies constitute an equilibrium, it suffices to show the payoffs from using these strategies cannot be improved by deviating, in any possible contingency, once and then reverting to the candidate strategies. Consider the payoff to deviating from $A_2 = 0$ by accepting an x_2 match. It cannot be optimal to accept the match and then search, given that you revert to the candidate strategy $A_1 = A_2 = 0$. So we can set $S_2 = 0$, which means $rV_2 = x_2 + \sigma(V_0 - V_2)$, which implies Δ_2 is proportional to $x_2 - b$. Hence, $\Delta_2 \leq 0$, and $A_2 = 1$ does not improve your payoff, iff $x_2 \leq b$. Similarly, $A_1 = 1$ does not improve your payoff iff $x_1 \leq b$, which is not binding given $x_2 \geq x_1$. Hence, the *type D* equilibrium exists iff $x_2 \leq b$, as claimed.

Now consider a *type F* equilibrium, where $A_1 = A_2 = 1$ and $S_1 = S_2 = 0$. This implies

$$\begin{aligned} rV_0 &= b + \alpha\pi(V_2 - V_0) + \alpha(1-\pi)(V_1 - V_0) \\ rV_1 &= x_1 + \sigma(V_0 - V_1) \\ rV_2 &= x_2 + \sigma(V_0 - V_2). \end{aligned}$$

For this to be an equilibrium, we require $\Delta_1 \geq 0$, which can be seen from straightforward algebra to hold iff $x_2 \leq y_1$ where y_1 is defined above, and $\Delta_2 \geq 0$, which is not binding. We also require $\Sigma_1 \leq 0$, which holds iff $x_2 \leq y_2$ where y_2 is defined above, and $\Sigma_2 \leq 0$, which is not binding. Hence, this equilibrium exists iff $x_2 \leq y_1$ and $x_2 \leq y_2$, as claimed.

The *type U* and *type P* equilibria are symmetric, in terms of the algebra, since in each case $A_1 = A_2 = 1$ and $S_j = 0$ for one of the two types of matches. The *type U* equilibrium implies

$$\begin{aligned} rV_0 &= b + \alpha\pi(V_2 - V_0) + \alpha(1-\pi)(V_1 - V_0) \\ rV_1 &= x_1 + (\sigma + \alpha)(V_0 - V_1) + \alpha\pi(V_2 - V_1) - d \\ rV_2 &= x_2 + \sigma(V_0 - V_2). \end{aligned}$$

The relevant inequalities can be shown to hold iff $x_1 \geq b + d$ and $x_2 \geq y_3$, as claimed. Similarly, the relevant inequalities for the *type P* equilibrium can be shown to hold iff $x_2 \geq b + d$ and $x_2 \leq y_4$.

Finally, consider a *type C* equilibria. This is a little more complicated because, although we know $A_1 = 0$, $A_2 = 1$ and $S_2 = 0$ in a *type C* equilibria, we need to specify S_1 differently depending on parameters – i.e., even though no one accepts an x_1 match in equilibrium, it matters what agents believe about S_1 off the equilibrium path. First, the value functions satisfy

$$\begin{aligned} rV_0 &= b + \alpha\pi(V_2 - V_0) \\ rV_1 &= x_1 + (\sigma + S_1\alpha)(V_0 - V_1) \\ rV_2 &= x_2 + \sigma(V_0 - V_2). \end{aligned}$$

Now suppose we set $S_1 = 1$ if $x_2 \geq y_5$, $S_1 = 0$ if $x_2 \leq y_6$, and $S_1 \in (0,1)$ if $x_2 \in (y_6, y_5)$, where

$$\begin{aligned} y_5 &= \frac{(r+\sigma+\alpha\pi)}{\pi(2\alpha+\sigma+r)}x_1 + \frac{\alpha\pi-(1-\pi)(r+\sigma)}{\alpha\pi(2\alpha+\sigma+r)}b + \frac{(r+\sigma+\alpha)(r+\sigma+\alpha\pi)}{\alpha\pi(2\alpha+\sigma+r)}d \\ y_6 &= \frac{(r+\sigma+\alpha\pi)}{\pi(\alpha+\sigma+r)}x_1 - \frac{(1-\pi)(r+\sigma)}{\pi(\alpha+\sigma+r)}b + \frac{(r+\sigma)(r+\sigma+\alpha\pi)}{\alpha\pi(\alpha+\sigma+r)}d. \end{aligned}$$

Given S_1 is set in this way, it is a matter of algebra to verify that all the conditions for the *type C* equilibrium hold iff $x_1 \leq b + d$ and $x_2 \geq y_1$. This completes the proof. ■

Proof of 2: To begin, insert V_i and N_i, as well as $A_2 = 1$ and $S_2 = 0$, into (3), and rearrange to express the objective function in terms of A_1 and S_1:

$$rW = \frac{\left\{\sigma\alpha S_1[2-(1-\pi)A_1]+\sigma^2\right\}b+\sigma\alpha(1-\pi)A_1(x_1-S_1d)+\alpha\pi(\sigma+2\alpha S_1)x_2}{\sigma\alpha(1-\pi)(1-S_1)A_1+(\sigma+\alpha\pi)(\sigma+2\alpha S_1)}. \tag{19}$$

If the planner chooses the *type C* strategy, then

$$rW = \frac{\alpha\pi x_2 + \sigma b}{\alpha\pi + \sigma}.$$

If he chooses the *type F* strategy, then

$$rW = \frac{\alpha(1-\pi)x_1 + \alpha\pi x_2 + \sigma b}{\alpha + \sigma}.$$

And if he chooses the *type U* strategy, then

$$rW = \frac{\alpha\sigma(1-\pi)[x_1 - (b+d)] + (\sigma + 2\alpha)(\alpha\pi x_2 + \sigma b)}{(\sigma + 2\alpha)(\alpha\pi + \sigma)}.$$

It is now a simple matter of solving for the parameter values such that each choice yields the greatest value of W. ■

Proof of 6: As a preliminary step, we take $G(x)$ as given, and observe that the number of unmatched agents evolves according to:

$$\dot{N}_0 = (1 - N_0)\sigma + (1 - N_0)G(Q)[\alpha + \alpha F(R)] - N_0\alpha[1 - F(R)].$$

This says that the flow into the set of unmatched agents is the number of matched agents who suffer an exogenous separation plus the number of matched agents who are abandoned by their partner or meet someone with $x < b + d$, while the flow out is the number of unmatched agents who meet someone with match value above $b + d$. Setting $\dot{N}_0 = 0$, we have

$$N_0 = \frac{\sigma + G(Q)\alpha[1 + F(R)]}{\sigma + G(Q)\alpha[1 + F(R)] + \alpha[1 - F(R)]}. \tag{20}$$

We now derive $G(x)$. First note that $G(R) = 0$. Next, denote the measure of set of agents who are matched with match quality $x \leq \bar{x}$ by $\mu(\bar{x}) = (1 - N_0)G(\bar{x})$. For $x \in [R, Q]$, this evolves according to

$$\begin{aligned}\dot{\mu}(x) &= N_0\alpha[F(x) - F(R)] + (1 - N_0)[G(Q) - G(x)]\alpha[F(x) - F(R)] \\ &\quad - G(x)(1 - N_0)[\sigma + 2\alpha + \alpha F(R) - \alpha F(x)].\end{aligned}$$

For $x > Q$, it evolves according to

$$\begin{aligned} \dot{\mu}(x) &= N_0\alpha[F(x) - F(R)] - (1 - N_0)[G(x) - G(Q)]\sigma \\ &\quad -G(Q)(1 - N_0)[\sigma + 2\alpha + \alpha F(R) - \alpha F(x)]. \end{aligned}$$

Now insert (20) into $\dot{\mu}(x) = 0$ and solve for the steady state $G(x)$, for any $x \geq R$, as a function of $G(Q)$:

$$G(x) = \begin{cases} \dfrac{[F(x) - F(R)][\sigma + 2\alpha G(Q)]}{[1 - F(R)](\sigma + 2\alpha)} & \text{if } x \in [R, Q] \\ \dfrac{\sigma[F(x) - F(R)] + 2\alpha G(Q)[1 - F(x)]}{\sigma[1 - F(R)]} & \text{if } x > Q \end{cases} \tag{21}$$

We can solve for $G(Q)$ by setting $x = Q$ in (21) and rearranging:

$$G(Q) = \frac{\sigma F(Q) - \sigma F(R)}{\sigma[1 - F(R)] + 2\alpha[1 - F(Q)]}. \tag{22}$$

Then we can substitute (22) into (21) to arrive at (13). Similarly, we can substitute (22) into (20) to arrive at (12). This completes the proof. ■

Proof of 7: To reduce notation slightly, we define $\bar{W} = (\sigma + 2\alpha)rW$ and maximize $\bar{W}$. The first order condition with respect to R is

$$\frac{\partial \bar{W}}{\partial R} = \frac{(b + d - R)\,\sigma\alpha F'(R)}{\sigma + \alpha - \alpha F(Q)} = 0,$$

which immediately implies $R = b + d$. The first order condition for Q can be written, after simplification,

$$\frac{\partial \bar{W}}{\partial Q} = \frac{\bar{W} - \sigma(b + d) - 2\alpha Q}{\sigma + \alpha - \alpha F(Q)}\alpha F'(Q) = 0,$$

which leads to (18) by straightforward algebra. Notice that

$$\frac{\partial T^o}{\partial Q} = 2\alpha\,[\sigma + \alpha - \alpha F(Q)] + \alpha\sigma d F'(Q) > 0.$$

Hence, there is a unique Q satisfying this condition. To check that it constitutes a maximum, we check the second order conditions. After simplification using the first order conditions, the Hessian matrix conveniently reduces to

$$\begin{bmatrix} \frac{-\sigma\alpha F'(R)}{\sigma+\alpha-\alpha F(Q)} & 0 \\ 0 & \frac{-2\alpha^2 F'(R)}{\sigma+\alpha-\alpha F(Q)} \end{bmatrix}.$$

Hence, the second order conditions hold, and the proof is complete. ■

Proof of 8: We know the efficient Q is the solution to (18). We also know that Q is an equilibrium if and only if it is in the interval $[\underline{q}, \overline{q}]$, where $\underline{q}$ is the solution to $q_0(Q) = Q$. In the limiting case $r = 0$, $q_0(Q) = Q$ can be rewritten $T^e(Q) = 0$, where

$$\begin{aligned} T^e(Q) &= \alpha[\sigma + 2\alpha - 2\alpha F(Q)]Q - \alpha\sigma F(R)b + \sigma[\sigma + 2\alpha - \alpha F(R)]d \\ &\quad -\alpha\sigma \int_R^Q x dF(x) - \alpha(\sigma + 2\alpha) \int_R^\infty x dF(x) \end{aligned}$$

after integrating by parts to replace $\int(1 - F)dx$ terms with $\int xdF$ terms. Observe that T^o and T^e are both increasing in Q, and that given $R = b + d$,

$$T^o(Q) - T^e(Q) = \sigma\alpha(Q - R).$$

Hence, T^o lies above T^e for all $Q > R$, and therefore $\underline{q}$ exceeds the planner's solution. Hence, any equilibrium Q exceeds the planner's solution. ■

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